AN ESSAY

ON

THE RELATION OF THE SEVERAL PARTS

OF A

MATHEMATICAL SCIENCE

TO

THE FUNDAMENTAL IDEA THEREIN CONTAINED;

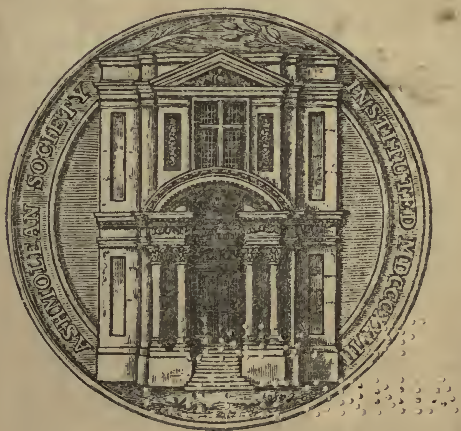
THE SUBSTANCE OF WHICH WAS READ BEFORE

THE ASHMOLEAN SOCIETY ON THE EVENING OF MAY 14, 1849.

BY

BARTHOLOMEW PRICE, M.A.

FELLOW AND TUTOR OF PEMBROKE COLLEGE, OXFORD.



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On the relation of the several parts of a Mathematical Science to the fundamental Idea therein contained.

THE inquirer into the principles and structure of a mathematical science has two questions to discuss: of which, one has reference directly to the source whence its scientific truth springs, and incidentally only to the history of the rise and growth of the particular science; and the other to the arrangement and sequence or connexion with each other of the different portions of the subject-matter of the science in the more perfect and developed form in which it is offered to his contemplation: the former of these two questions belongs to Metaphysic, and the latter to Logic; but they are so intimately related to each other, that it is impossible to consider either, without introducing matter which more properly belongs to the other: and although the object in the following paper is to elucidate the latter, viz. the logical question, yet the assumption of certain propositions, the discussion of whose certitude and truth belongs to metaphysic, will be necessary; but it is to be borne in mind, that they have been assumed in order to enable us to follow out our train of argument in the particular subject which we are going to discuss.

The "vexata quæstio" of metaphysic, from the rise of philosophy to the present time, is this, "Have we, or have we not, any ideas which are necessarily and universally true?" and it resolves itself into a question of another form, "Have we, or have we not, any knowledge independent of experience?"

On these questions two schools have been formed; one declares that all knowledge is given in experience, and that its materials are derived from *sensations*, which are impressions on the mind of external objects; and that the mind, endowed as it is with the faculties of attention and retention, is the repository of them; and that from reflection on these, combination of them, and generalizations founded on them, knowledge, real and systematic, is formed: the other school declares, that experience gives us only a portion of our knowledge; that there are elements of it which were never derived

from sensation, and which, on account of their necessity and universality, absolutely transcend all experience; that the rough materials are given by empirical observation, and that they may be accumulated in any quantity and quality; but that the *form* which is imposed on them, and which constitutes them systematic knowledge, is supplied by the mind: thus, that systematic knowledge consists of two elements, of which one is acquired from experience, and the other from certain *ideas* natural to the mind, which perhaps observation and experience may call into action; but that the knowledge arising from these has such characteristics of necessity and universality which no amount of experience can possibly give.

While then, of these two contradictory schools, one says that all knowledge is derived from experience, and passes through the senses to the mind, which is the storehouse of it; the other assumes that there are certain ideas in the mind previous to all contact with the world of sense, which are sources and mainsprings of knowledge, and that whatever we know of external objects, we know only so far as they accord with these ideas, and as far as these ideas combine them: one therefore says that all real and systematic knowledge comes from within; the other, that it comes from without: one, that all knowledge is subjective; the other, that there are certain real objects existent independently of the human mind, which the mind has the capacity of cognizing, and that when, and only when, these are cognized, has it real and substantial knowledge: the one says that our knowledge starts with ideas, and ends with facts that accord with these ideas; the other, that it begins with facts, and by a plastic faculty of the human mind, which combines many facts in one idea, ends with the idea; and while one school would sum up its philosophy in an axiomatic dictum of the scholastic doctors, "nihil est in intellectu, quod non fuerit in sensu," and almost ignore the restriction of Leibnitz, "nisi ipse intellectus," the other school would reply by such arguments as these; "You may take away from your experience-conception of a body all that is empirical therein, colour, hardness, softness, weight, this or that shape, still the *space* remains which is occupied, and *that* you cannot take away: or you may omit from your empirical conception of any corporeal or incorporeal object all that experience teaches, yet you cannot take away from it that by which you think upon it as a *substance;* and thus it would argue that these and such conceptions must have their seat in the faculty of cognition *a priori*."

An examination of the logical processes and arrangement of some of the mathematical and physico-mathematical sciences will involve

the adoption of one or the other of the two contradictory systems. It is beside the object of the present paper to discuss their respective arguments; but the four following propositions contain the substance of what is assumed and will be said; and from these it is apparent which of the two systems we have adopted, and the illustrations, as far as their value goes, will give evidence to the truth of it.

I. There are in the human mind certain general ideas or conceptions, capable of development, such as those of number, space, time, motion, substance, and the like.

II. The pure and exact sciences consist of systematised series and aggregates of the several processes by which these pregnant ideas are called into action and subsequently developed.

III. That although we cannot *a priori* assert that these sciences are applicable to objects of the external world, and to the explanation of phænomena which we observe, yet they may be so, and are so, in so far as we can prove that the primary and axiomatic parts of the idea have their counterparts in nature.

IV. That thus every such science to be useful must consist of two parts; one in which the idea pregnant with consequences is analysed and developed; and a second, in which the phænomena of nature are observed with the view of determining, first, whether at all, and, secondly, how far, the laws of the external world accord with the axiomatic statement of the science.

Without entering into the question whence the mind gets its ideas, whether they are innate, or whether they are acquired; whether they are the necessary results of a comparison of the objects of the external world, or whether the mind does not of necessity get them so far as to express them by words from the circumstances in which itself grows and is a mind; yet independent of such an investigation, in the construction of a science which arises from such an idea as I conceive to be at the foundation of the purely mathematical sciences, there must be two logical processes; one, the deductive, by means of which parts and properties of the idea are enunciated and defined, divisions and classifications are made, what is arbitrary and accidental is separated from what is necessary and germinal, by means of which propositions are formed from the comparison of parts of the idea which are more and less general, and the equivalents of such propositions are framed, and from general propositions others less general are deduced; that process, in short, by which we resolve the pregnant idea into its several constituent parts, and deduce particular propositions which

the general statement of the idea imports; but when our deductive science has been thus constructed from an idea in the mind, our next business is, to inquire, whether our scientific idea and propositions have their counterparts in the external world of sense; our science may be merely an intellectual discipline, useful perhaps for cultivating the mental faculties, and indicating the import of pregnant propositions, and no more than this. Or, since man is not like one of Aristotle's deities αὐτάρκης, happy in purely subjective contemplation, but inasmuch as he has body as well as soul, and lives in contact with a marvellous complication of external phænomena, to the examination of which he is driven by mental impulses, it is likely that the external world corresponds to his subjective ideas; at all events it is worth while to inquire whether it does: hence arises the necessity of examining facts and phænomena, with the view of detecting general laws subject to which they take place, in order that we may compare them with the propositions, axiomatic or inferred, of our science; we must therefore analyse effects with the object of determining the laws of their necessary antecedent causes: hence there must be another process, which shall guide us in gathering from our experience of and intercourse with the external world the general laws subject to which its agencies are conducted; which shall give us rules to ascend from particular and observed facts to general propositions; which shall furnish us with criteria to discover the general law lying concealed in the particular instance; by this process we shall interrogate nature, and determine how far her laws are identical with those of our scientific idea; this process is manifestly the reverse one to the former, and is called the inductive; and although we might *a priori* (on account of the manner in which, as metaphysicians say, the idea is arrived at) expect that our science would correspond with the phænomena of nature, yet we can apply it only so far as the inductive propositions allow us; this latter process then will in a manner make real and substantial that science which before was only subjective, being a creature of and existing in the human mind; in the complete discussion therefore of any exact science which has an objective reality as well as subjective existence these two processes must enter.

If this be a true account of the constitution of a science, the division of sciences into "pure" and "applied" is founded on a wrong principle; for the more correct division would be into sciences "perfect" and "imperfect:" those, that is, which have reached a state in which the above two elements are combined, and those

which want one or the other of them. By perfect sciences therefore I mean such as have a leading and pregnant idea, supplied by the mind; an idea distinctly apprehended, clearly defined, and the limits of it accurately set out: that which renders the truths of the science superior to all experimental evidence, and gives them an universality and a necessity which is characteristic of the science; such are, I conceive, the ideas of *number, geometrical space, motion*, and such like, and therefore the several corresponding sciences of *number, geometry, mechanics*, ought to have the corresponding notes of perfection; these however have an application of greater or less latitude in the phænomena of the external world; greater or less, I say, because the *idea* has certain phases and the *a priori* science certain parts, the counterparts to which nature does not exhibit: though she therefore may illustrate and interpret much, yet there may be and is much which goes beyond her[a]; although probably she has first *suggested* the idea which otherwise would lie dormant, still the mind, by a salient and plastic power of its own, has given to the idea such powers, that it is no longer to be kept within the limits which nature and experience impose, but going beyond these constructs a science; and such sciences have in many cases foretold, anticipated, and led the way to some of the most famous discoveries of natural philosophy, results have been deductively inferred from them which have taken the world by surprise, and an examination of nature has subsequently shewn her exact accordance with them: assuming then the principles of the science to be true, all consequences, those analogous to nature's laws no more than others, follow with the same logical cogency; and the deductive processes of our sciences of number, geometrical space, and motion, are exactly the same for all abstract properties of number, for the most recondite investigations of infinitesimal analysis, for all conceivable geometrical surfaces, and for all laws of dynamical force, as they are for the applied branches

[a] A late eminent Professor of Moral Philosophy in this University, Mr. Mills, writes as follows in a lecture on the Origin of Ideas: "We may safely allow that sensation gives the first impulse; we may agree with Bacon and Locke that knowledge is built upon experience; but it is the active and independent power of the understanding which regulates and fashions anew the information communicated by the feelings of sensation, and which ascends from the first lessons of experience to the general and immutable principles of virtue and science. By this divine light of reason kindled in the soul, man vindicates his high original and future destiny; developes the faculties and energies with which his Creator has endowed him; and, so far as a humble sense of his dependence on the Fountain of all intelligence will permit, feels a just pride as he contemplates the moral and intellectual strength of Butler, Pascal, or Newton."

of these several sciences: for commercial arithmetic, for land-measuring, for physical astronomy.

It is true that in a particular science one element may appear more prominent than the other; in our science of mechanics the experimental and applied part at once arrests the attention, for the whole world is in motion, matter's state is continually changing, for momentum is ever being transferred from one body to another, and thus in most cases we fail to discover the pure science of motion or to realize its idea—it is lost in its applied form; yet not the less really does it exist, and a science of motion may be drawn from it: it is on this principle that D'Alembert has constructed his "Traité de Dynamique;" so on the other hand, the science of number seems to us now to be the necessary development of the abstract idea of quantuplicity; we do not seek for any confirmation of its truths in experience, we scarcely look beyond the range of the abstract science for illustration, so easily apprehensible do we think its truths to be; thus in common arithmetic, as soon as the idea of definite number is apprehended, we do not even ordinarily think it necessary to state such axioms as "If equal numbers be added to equals the wholes are equal," "Numbers equal to the same number are equal to one another," or to explain what equality of numbers is; and the common rules of arithmetic, those of addition, subtraction, proportion, &c. are at once assented to; and these become the first premises of our science, which are immediately deduced from the idea, and from which all the results of the science follow by a deductive process; yet there is an applied part of the science which renders our common arithmetic useful: commercial arithmetic, our systems of weights and measures, division of time, exchanges, are all applications of the rules and processes of the exact science; but before this can be logically done, we must have assured ourselves by an inductive process, which however may be so rapid as to escape observation, that the axioms and laws of our pure science have their counterparts in the applied: as for instance, that matter is discretive and susceptible of discontinuous, though of definite, division; that it is additive simply, and that when two units are added, one is not absorbed into the other: these and such like properties of the pure science must also be true in the matter of the applied science; doubtless these two elements may be best traced in the history of the science's growth, and in the case of common arithmetic we are under a disadvantage, for little or nothing is known of its history; in the earliest records it is found to be in its principles and rules nearly as perfect as it is now; and as its idea is so simple, and as it exists in only

one dimension, the mind apprehends it at once, and we are wont to overlook the inductive and applied element. In another branch however of the same science, viz. in Infinitesimal Analysis, we have more ample illustration, so that the applied element is more clearly apprehended than the ideal; the conception of number is at the foundation of it, but under other forms than those with which we conceived of it in arithmetic, viz. under the forms of infinite discretiveness and of continuousness; for whereas in arithmetic the idea of number is of it in certain *determined* proportions, and in algebra of certain *undetermined*, and in both of these in *finite* proportions, so in infinitesimal analysis our conception is of number in infinitely small, and *therefore* in infinitely large proportions; and whereas again in arithmetic and algebra we pass "per saltus" from one number to another, and neglect all intermediate numbers, so in infinitesimal analysis we conceive of number as increasing by continuous growth; discontinuousness and finiteness are properties of number in arithmetic and algebra; but in the infinitesimal calculus continuousness and infinity are parts of the pregnant idea, and which are developed into the science; and the conception of it under such peculiar phases authorizes us to define infinitesimals and their orders, to construct rules for determining them, or, in other words, to create the subject materials of the science, to invent a convenient nomenclature, to enunciate axiomatic laws respecting them, such as the following: "a finite number of infinitesimals of a given order makes no appreciable increase when added to finite numbers, or to infinitesimals of a lower order;" and to deduce from them all the consequences which necessarily follow. Such is the method of infinitesimal analysis, and the assemblage of the processes above described constitutes the Differential Calculus. In the results we are led to such simple formulæ, which are so immediately applicable to questions of geometry and mechanics (the subjects which originated the science) that the principles of the pure science are lost sight of; and although the science is as perfect as any, and the two elements distinctly are combined in it, yet the ideal has scarcely been recognised; but we know the history of it; how it presented itself in a confused and applied state, and under two different phases to its founders: how Newton realized the property of continuous growth, and Leibnitz that of infinity and infinitesimals; and how it has reached its present perfect state by successive improvements; and we know now that it has a sure and logical basis, on the strength of which it is not only applicable to the explanation and prediction of geometrical and mechanical

truths, but also such a due and exact conception has been formed of its idea, as has led to the most recondite results, as to the calculus of variations, the theory of definite integrals, and so on. It is to the ideal element that the labours of the most eminent analysts have been directed in the latter years; and if we may judge of the future by past success, we may hope that within a few years the integral calculus may be nearly as complete as the differential.

In geometry I may remark, that for the most part the ideal element prevails over the experimental.

The perfection of a science depends on the due adjustment of these two elements: this is evident to the historical inquirer into the growth of a science: for the most part, the experimental facts are first observed; but they exist singly, one by one; they require binding, or, in the apt language of Professor Whewell, there must be a colligation of them; a principle of unity is required, and such is the idea; but this lying hid under a mass of special applications of it, is but dimly, if at all, recognised at first, and thus the inductive process advances but slowly and often unsuccessively: observers may toil, and the result of their labours will be but a series of disconnected facts. When however, at last, by the sagacity of a master mind the idea is recognised, unity is given to all; all become parts and applications of the one idea; the chaotic observations become systematic knowledge, and thus a philosophical and arranged science is formed; and a science too which is not only commensurate with the facts observed, but of much wider applicability: for the idea which has been *generally* suggested by observation has been supplied by the mind, and the mind has given it properties of universality and necessity, with which no amount of experience could ever invest it; and thus it involves consequences of greater extent than those observations which called it from its source; and it becomes the germ of a pure science: and when it has been distinctly apprehended, particular phases of it are enunciated in precise language, accurate and distinct divisions of the several branches of it are posited, definitions are framed, and the consequences of the idea are traced by a deductive process from the axioms, which are the major premises of the first syllogisms: and a science thus constructed is ready henceforward to unravel complex phænomena which may occur under conditions consistent with its axiomatic laws; and also to foretel what will happen when such and such circumstances concur. The determination however of the consistency of these circumstances with the axiomatic laws requires a process different to the deductive of which an outline is given: for particular facts and cases

must be analysed; their accidents must be eliminated; their essential qualities examined, to detect whether at all, and how far, they accord with the philosophical axioms of the pure science; and we must endeavour in these instances to discover the connexion of necessary antecedent and consequent, of cause and effect, which lies hid in them: thus, for example, to anticipate what will be presently said, ere the exact science of mechanics can be applied to the explanation of the facts of Physical Astronomy, we must examine the *matter* of the universe in order to determine whether it has properties accordant with the laws of motion, which are the axioms of mechanics; and subsequently, as the exact science of motion will include all laws of force, we must by observation determine what the particular law is which prevails in physical astronomy; and the examination of facts with this object in view has led to the grandest instances of induction: and inasmuch as perhaps otherwise our science might be useful as an intellectual exercise, it hereby becomes an applied science; and thus, chronologically, the examination of external phænomena will enter twice into the science; first, in the formation of it, when the idea is suggested; and again, when external nature is examined in order to determine the agreement or disagreement of her laws with those of our *a priori* conception. A complete treatise on any perfect science ought therefore to consist of two parts, one in which the principles of the pure science are explained and its consequences deduced, and a second in which are discussed the inductive arguments and the inductive laws for the application of the pure science to the phænomena of nature. If therefore Logic be that science which treats of the laws of thought, and constructs rules for educing from given truths other truths which they contain, and with this object enables us to analyse conceptions, to examine their consistency or inconsistency, and affords tests whereby to judge of the cogency of an argument, and if the methods of investigation which it provides be applicable to all sciences and all man's knowledge, whatever be the source of it, logic, fully to discharge its office, must consist of two parts, and provide two processes: one the reverse of the other; one, that is, by which the special truths with which a more general proposition is pregnant are evolved, and the other by which we are enabled to discover a general proposition or law concealed in a particular instance; and if its character be thus universal, that is, if it be the process of educing truth from given truths, which either observation, or experiment, or conscience, or necessary attributes of the mind, give to us, we must not transform it into a kind of Hermeneutic or superior grammar, but we must exclude from it

all such questions as belong to metaphysic, rhetoric, or to psychology, and restrict it to its own province and subject matter, to the evolution of truth from other truths, whether by deduction or by induction, and to the consideration of conceptions, whether they be *a priori* or empirical, whether they be only in thought or expressed in words.

Whether any science has attained to the utmost perfection is a question which I suppose all will answer in the negative; but that some have reached that state in which the two elements are recognised and adjusted, so that they may fairly be called "perfect" in contradistinction to others which are still far behind them, will, I think, be allowed by all but the most bigoted empiricist: the question then of perfectness or imperfectness is one of degree; but amongst those which have attained to a degree of comparative perfection, I should mention that of Number in its three branches of common Arithmetic, Algebra, and Infinitesimal calculus; that of Geometrical Space, and of the particular form of the science called algebraical geometry; that of Motion "la Mécanique." Yet there are others which are on their way to this perfect state; the ideas of which have not yet been clearly apprehended, and the principles of classification not yet distinctly framed; but the materials for which in their applied and partial form exist in the external world; from which certain general propositions have been formed by an inductive process, but which are at present little else than observed uniformities, being general formulæ including in their grasp many particular cases, but yet are not statements of axiomatic properties of a central idea, fitted to be the nucleus and germ of a science; and therefore such knowledge, though it be to a degree systematised, does not reach that standard of perfection to which the three sciences above mentioned conform; the dictum "that fluids press equally in all directions," the idea of polarity which Professor Whewell makes to be the germ of the electrical and its kindred sciences, that of resemblance and analogy which enters so largely into the structure of systematic treatises on botany and natural history as the principle of classification, though they do colligate what would otherwise be unconnected facts, and thus give order where would otherwise be confusion; yet are not pregnant ideas of the respective sciences in the same way that number is of arithmetic, and space of geometry, and motion of mechanics; proofs of this assertion are evident from the difference of relation that the facts of the science bear to the above *ideas* according as new discoveries are successively made; ever since similar effects have been

found to arise from electrical action, whether developed by a machine, by a battery, or by a revolving magnet, the idea of polarity has entered with more distinctness of conception into an explanation of the phænomena of machine-electricity; and the principle of the systematic arrangement of plants has been lately founded on a consideration of their physiological characters instead of on their external flowers as is the case in Linnæus's arrangement; in some of the still-imperfect sciences the idea doubtless exists in the mind, though in a rough and indistinctly-conceived form; such as the idea of mechanics was before the age of Galileo; and such as perhaps the idea of the theory of undulations is at present; the existence of an etherial medium is assumed, the constitution of which however has not yet been so clearly conceived that its properties can be accurately stated; still however the science of optics is on its road towards perfection, and has already satisfied a searching test; that is, it has predicted phænomena which observation has subsequently shewn to exist; an evidence of its truth of the same nature as that which gives to the theory of gravitation its greatest certainty; viz. the discovery of Neptune, the calculation of solar nutation, the lunar disturbance owing to the earth's oblateness; we may venture therefore to foretel that ere long such an imperfect science will be ranked amongst the exact sciences, having an idea adequately realized, and its results accurately detailed.

As so much has been said above on the science of number in illustration of the relation to each other of the two elements of a science, it is unnecessary to add more; but I propose to consider the sciences of Geometry and Motion at greater length, and to shew how exactly the processes which I have attempted to describe hold good in these particular sciences. The *idea* of the science of geometry is *geometrical space;* the term cannot be defined, being too large in all that it implies to admit of being trammelled with words; and for this very reason it is, that it is pregnant with consequences. In the construction of the science, then, certain phases of the idea are first enunciated, and certain properties of it are stated which are necessary to an adequate conception of it; and if which were not, geometrical space would not be what we conceive it to be. These statements are our axioms, and the major premises of the first syllogisms from which we deduce the several specific propositions which enunciate properties of the leading idea. The axioms are such as, "Things equal to the same thing are equal to one another," and the following six axioms of the first book of Euclid. For all these, though true, perhaps, of every thing capable of measurement,

are yet, I conceive, enunciated in Euclid as properties of geometrical space only. So again are properties of space expressed in the following axioms: the conception of equal spaces in the axiom "Magnitudes which coincide with one another, or exactly fill the same space, are equal"—"The whole is greater than its part;" and the conception of ratio of spaces in the axiom in the fifth book—"Ratio is the relation of two quantities to each other in respect of (not position or colour or hardness, but) quantuplicity" (κατὰ πηλικότητα). These are true of all geometrical space, involved in any adequate conception of it, and without which it would not be what it is: they are truths too of such certainty as no amount of experience could ever give to them;—experience may suggest them, but it will give but a rough outline and a crude form of them; it will present to us a line nearly straight, and nearly without breadth, as for instance the joint of two boards of different colours: but there are unevennesses which the mind must abstract, and thus provide for itself the conception of a line perfectly straight, and lying evenly between its extreme points. In a similar manner are our conceptions of a perfect sphere and a perfect ellipse formed: the mind then acts on these imperfect objects, and gives them a necessity which is peculiar to its own constitution; and the enunciation of such axioms is the first step in the philosophical construction of a science. Again, the idea is such that space is tridimensional; hence arises a perfect division of the subject into three parts, corresponding to space of one, to space of two, and to space of three dimensions. Of space of one dimension or lines we have a twofold division of lines, curved and straight; and of curved lines we have several species of circles, ellipses, cycloids, curves of double curvature, and so on: and we have figures formed of combinations of straight lines, as e. g. triangles, parallelograms, trapeziums; and we have parallel straight lines; and thus there is the most perfect division possible of the several specific forms which the pregnant idea involves, and a most perfect subordination of classes from even summum genus down to infima species; but these many divisions having been made, it becomes necessary to explain the meaning of the terms we employ; hence our need of definitions, which are definitions of the several *classes;* not the same as the axioms, inasmuch as they are not pregnant with consequences, and are not used as premises in any syllogism by means of which we deduce new properties from old ones; but as many of the specific forms of the original idea involve consequences peculiar to that particular species, it is necessary to enunciate axioms of rather a different kind, which shall state pro-

perties, of not the idea in all its generality, but of certain particular species of it,—such as the axiom about straight lines—"Two straight lines cannot enclose a space;" that about right angles—"All right angles are equal;" that about parallel straight lines, and so on; all these differ from the definitions, inasmuch as the definitions are definitions only of the words, and do not involve any of the properties of the things whose names they explain. But axioms, specific though they be, are replete with all the consequences which are involved in the particular forms of geometrical space corresponding to them. By help of the axiom "Two straight lines cannot enclose a space," the fourth proposition of the first book of Euclid is proved; but the proof of no proposition rests on the definition that "straight lines are those which lie evenly between its extreme points;" for whereas the axiom enunciates an essential property of such lines from which all other properties of lines may be educed; the definition gives little else than a synonym of a straight line. Hence also arises the necessity of a formal statement of some property of parallel straight lines besides the definition of them. Here also, did the limits of the paper admit of such extension, we might shew how most of the properties of space of two dimensions depend on the axiom concerning equality of magnitudes filling the same space; how exact the division is of figures, according to their bounding lines, whether straight or curved; and of surfaces into plane and curved: and so we might proceed to discuss as parts of the same science of space those results, which are involved in our conception of it as a quantity infinitely discretive and continuous, whereby we shall be supplied with principles for defining a point as the inferior limit of space, a straight line as the superior limit of a circle's arc (when the radius is infinitely large), and a plane as the superior limit of a spherical surface, and parallel straight lines as the sides of a triangle whose base is finite and vertex at an infinite distance. All these properties and innumerable others are involved in the general idea of space, from which, according to the rules of deductive logic, they are to be deduced: the general idea is thus to be separated into its constituent elements; and the same logical process is followed in all cases; that is, it is indifferent to the positive science, whether the curves and surfaces have their counterpart in nature or not; whether we are discussing the properties of the more complicated curves, of cycloids, of lemniscates, surfaces of elasticity, or of ellipses, triangles and spheres, in all cases the same rules are followed, and the same axioms are the first major premises. But, on the other hand, if we intend to apply our geometrical figures and

their properties to the explanation of cosmical phænomena, that is, if we intend to make our pure science an applied science, then we must shew from observation and experiment that the fundamental principles and axioms are true in the matter to which we are to apply their consequences: here then an inductive examination of facts enters into the science; general laws to which they are subject must be discovered: as for instance, in the application of geometry to questions of astronomy, we must first assure ourselves that such a motion as the planets are assumed to have is consistent with the axioms of geometry; and with this object we have to connect our conception of a continuously-moving particle with that of geometrical space capable of infinite discretiveness and of continuousness; and again by an accurate observation of the planet's position night after night, and noting down its successive places, we shall conclude that its orbit is an ellipse; and when this has been done, we shall be authorized to apply to the planet's motion all the properties of the ellipse; our pure geometry will then come in, and by virtue of it we can enunciate certain properties of such elliptic orbits, which will go far to test the truth of our observations, and to foretel certain results of such a motion. Such has been the course of astronomy; for had not the Greek geometers clearly apprehended, and by their deductive processes accurately analysed the idea of geometrical space, and thereby fully discussed the properties of the conic sections, the conception of an ellipse would not have been ready to Kepler to simplify the complicated theory of epicyclical motion, and Newton might not have been able by his mechanics to deduce from the ellipse his law of gravitation; and had not a positive science of algebraical geometry been previously constructed, Fresnel might not have been able to express his surface of elasticity and wave surface; and sir W. Hamilton might not have been able to discover the cuspal points whereby he was led to the phænomena of conical refraction.

The other science, the principles of the structure of which I propose to consider, is Mechanics, "la Mécanique," which I should prefer to call the science of Motion; for motion is its pregnant idea. "Le mouvement et ses propriétés générales," D'Alembert writes in the Discours Préliminaires, "sont le premier et le principal objet de la mécanique;" what motion is in itself is a metaphysical question, the consideration of which does not fall within the scope of the present paper: sufficient for us that we are able to state such affections of it as may be the axioms of the science, and the basis of our reasoning; in the first place, we do not conceive of motion, except of something moving, and that something we call *matter;*

motion must be clothed, and matter is that wherein it is clothed, and in which it consists. "In mechanics," says Professor Whewell, "we know of matter only as the subject on which force acts:" this matter however of the abstract science is not necessarily what is sensible and what gravitates and is heavy, for by matter we mean none other than that which moves, and in which motion may reside: thus the moving molecules of the etherial medium are such matter as the science of motion recognises. The *ultimate facts* of the science are *motion* and *matter*. Motion of matter also involves two conditions; it takes place in *space* and *during time; matter*, we say, exists in space, and such existence is called *extensiveness;* and inasmuch as we do not conceive of two different particles of matter occupying the same space at the same time, we say that it has also the quality of impenetrability. *Time* again enters into our conception of matter in motion in two ways: we do not conceive a particle of matter to be in two different places at the same time, and we do not conceive of it passing from one position to another instantaneously; time must be occupied in the passage. Such are the four ideal elements on which and on their relation to each other the science of motion is raised;—motion, matter, space, time. We treat of matter in motion, and of motion as an affection of matter; and motion takes place in space and during time. From these elements arises our conception of velocity, which is the degree of swiftness or slowness with which matter changes its position. Velocity, we say, is greater, the greater the space passed over in a given time, and the less the time spent in passing over a given space; and thus we may define velocity, after the manner of most English mechanicians, to be, according to the laws of variation, the ratio of the space passed over to the time to which it is due, or after the manner of most continental writers, to be the space passed through in an *unit* of time. In the case however of variable velocity, the circumstances adapt themselves to the principles of the infinitesimal calculus; and under the different aspects in which it is considered we should define velocity to the ratio of the increment of the space to the increment of the time to which it is due; or we should reduce the unit of time, so that it should become an infinitesimal. In either case the determination of the finite velocity generated in a finite time becomes a problem of the integral calculus. When matter however moves with an increased or diminished velocity, a question arises whether this change is due to an external cause acting on it, or to an intrinsic power of its own to affect its own state. The old Aristotelian philosophy concluded, that cosmical matter at least had

the latter property. Galileo first enabled mechanicians to raise their science on an axiom embodying the former alternative: matter, he said, is *inert*, it has no intrinsic power to change its own state, whether that be of rest or motion: if it be at rest, it will remain at rest, and if it be in motion, it will continue to move; and to move without any increase or diminution of velocity; whatever action matter may exert on other matter, it has no power of acting on itself; hence when matter's state changes we are authorized to seek for the cause of the change in some source external to it; and in an adequate source, inasmuch as it will neither absorb into itself, nor generate of its own resources any change of its state. This axiom then of matters *inertia* is of the greatest importance to the construction of the science, for by it the equations of motion, which are the first propositions of the science, are formed, and as follows,—when matter's velocity is changed, the *expressed* or *developed* change of velocity is exactly equal to that *impressed* on or *communicated* to it under the following conditions: motion is conceived to *reside in* and to be *of*, matter; and matter is conceived to be measurable, that is, according to the principles of number, one quantity of matter is any number of times another quantity: and thus if we take a certain quantity of matter as the unit, any other may be expressed by a certain number of this unit; but this requires further explanation: we have spoken of matters existing *in* and occupying space, and this property we have called *extensiveness;* but a greater or less quantity of matter may be contained in the same space, according as it is packed more closely or rarely together; it may be more or less dense; such a property we call *intension*, and we speak of matter as more or less intense according as a greater or less quantity is contained in a given space: hence the quantity of matter varies both as to the space it occupies, and as to its density; and therefore according to the principles of number, any given quantity of matter will contain a certain number of particles of matter, by being compared with a given unit of given extension and given intension; the mode of comparison will be evident from what follows: when therefore any quantity of matter, or a given mass, as it is called, is moving with a given velocity, each unit particle moves with this velocity, and therefore if the particle had been at rest, and afterwards is moving with this velocity, (both of which suppositions are in accordance with the principles of the science,) the velocity must have been communicated to each unit particle from some external source, and therefore, by the law of inertia, the whole transferred velocity will be as many times the ve-

locity of each unit particle, as there are particles in the body; hence, therefore, if the velocity of one particle be symbolised by v, the whole velocity which has been transferred to the whole body will be (if m represents the number of particles) mv: this expression, which enters so largely into our science, we call *momentum*, or *quantity of motion*, and we consider it transferable from one mass to another, and to be such that none is lost in the transfer; so that we may now modify what was said conditionally of quantities of matter being equal both intensively and extensively, and unconditionally enunciate the following axiom: when matter's momentum is changed the *developed* or *expressed* momentum is exactly equal to the *communicated* or *impressed momentum*: this axiom when mathematically expressed forms the equation of motion, and being capable of expansion, is the major premiss from which all the results of the science deductively follow; if therefore there be a known mass m moving with a certain velocity v, its momentum is equal to mv, and if the whole of its momentum be transferred to a mass m' at rest, but which subsequently moves with a velocity v', then we have

$$mv = m'v';$$

hence if we knew v, and v', (and our ideas of space and time enable us to determine these,) we can compare m and m'; and it is to be observed that this is the mode we adopt for weighing masses.

In the case in which momentum is not instantaneously but gradually transferred from one mass to another, the successively transferred elements of momentum are equal to the successively developed ones; and as the mass is unaffected by lapse of time, it remains the same, and the equation of motion in this case becomes

$$\text{Mass} \times \text{impressed element of velocity} \\ = \text{mass} \times \text{expressed element of velocity.}$$

The limits of the paper do not allow me to enter into detail on the method of resolving and compounding velocities; but I may remark, that as soon as we have determined that velocity is to be expressed by $\frac{ds}{dt}$, the subject resolves itself into a question of *number*; and the laws of resolution and composition follow from the equation

$$ds^2 = dx^2 + dy^2 + dz^2.$$

As an example of the mode in which this equation is pregnant with consequences, I will take the following: since the velocity expressed

along the path of the moving particle is represented by $\frac{ds}{dt}$, we have according to Maclaurin's method of resolution

$$\frac{dx}{dt} = \text{expressed velocity along the axis of } x,$$

$$\frac{dy}{dt} = \quad .. \quad .. \quad .. \quad .. \quad y,$$

$$\frac{dz}{dt} = \quad .. \quad .. \quad .. \quad .. \quad z;$$

and therefore $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ are the several increments of the resolved expressed velocities due to an unit of time; if therefore there is no impressed velocity, each of these quantities is equal to zero; and therefore integrating twice, and adding constants

$$\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma} = t,$$

the first three of which equalities are the equations to a straight line, whence we conclude, that if a particle moves, and without any velocity being impressed on it, it moves in a straight line.

The general idea of the science being that of *motion*, it is of motion either in act or in power, either actual or virtual: in the latter case the science becomes that of statics, and its principle, by the equality of impressed and expressed momenta, that of virtual velocities; from which result the six equations of equilibrium; hence the problem of motion of rigid bodies is brought within the range of the science by means of D'Alembert's principle, and likewise the assemblage of propositions which forms the science. The primary equations of motion then having been formed, all the results follow by a deductive process from them; they involve all the consequences of the science, and however particular the last propositions be, and however far removed from the first axiom, yet the axioms exist in them, more or less near to the surface, and are capable of being drawn out; and the axioms are so general that they include all laws of transferred momenta, and all results follow with a logical cogency equally valid[b].

[b] It will be observed that in the preceding sketch, nothing has been said of "force," the idea of which enters so largely into most of our text books, that on it the whole science is raised; the word has been purposely omitted, because, first, force seems to be synonymous with mechanical *cause*, and therefore if its conception be the fundamental idea, the business of the science will be to

But after the pure science of motion has been evolved from the pregnant idea, can we apply it to the explanation of phænomena of external Nature? Not until we have established that the matter of Nature has properties which are the counterparts of the axioms of the science, and that the matter moves according to a law which the principles of the science include. With the object of determining these two points, we must examine and analyse Nature and her matter, and by an inductive process conclude what her laws are: first then we must inquire whether Nature's matter is moveable, impenetrable, and inert; whether it is in accordance with the laws of composition of velocities, and of momentum. The first four of these properties are proved inductively by the instances and experiments cited in the ordinary text books, and into an examination of which I need not therefore enter: the law of momentum is demonstrated by Attwood's machine, whence we learn that the increment of momentum due to any short time dt is equal to the product of the mass moved, and the increment of the velocity due to the same time dt. Such experiments prove the truth of the laws, axioms in the case of the matter of the earth; and then, by an inductive extension of such properties, we assume them to be true of the matter of the whole material universe; that is, we extend to the sun and to the other members of the solar system properties of matter which we have proved to be true of only one of the secondaries, viz. the earth: and our extension too goes farther than this, inasmuch as the law, subject to which momentum is transferred from one of the

discuss the relation between *cause* and *effect;* and thus those writers who have adopted this view, and stated its principle philosophically, enunciate as axioms the mechanical translations of such propositions, as, "there is no effect without a cause," "effects are produced by adequate causes," and speak of the composition of causes; but the equations on which the whole science is raised state the equality of momentum transferred and momentum developed which are exactly the *same* thing under two different aspects, and cannot with justness of language be called *cause* and *effect;* the source of the communicated motion, whether it be the muscular action of a cricketer's arm, the earth's attraction, electrical action, and so on, may more properly be called the causes of the motion, and into these the science of mechanics does not enter; in its investigations it assumes the laws according to which such momenta are impressed, and equates such communicated motion to that which is expressed in the moving mass: secondly, the word *force* has been used in so many different meanings, that it is well to avoid an ambiguous expression which is constantly a stumbling-block to students; we mean by it roughly what produces or tends to produce motion, and then we apply it to statical force, dynamical force, impulsive force, accelerating force, moving force, labouring force, &c., &c., and by each term intend a different thing.

bodies to another, can be tested only in its limit at the earth's surface; but all these extensions are made in accordance with canons which an inductive logic supplies, and of the truth of them thenceforward the mind has no doubt, and these become the "laws of motion," or the axioms of the science in its applied form: but I would remark that such an inductive analysis of facts can never give to such axioms that necessity and universality which are the characteristics of the idea of the pure science; it may fit them to the purposes of physical astronomy, but cannot constitute them primary laws of the exact science. Such an examination however fulfils its object; though it does not render the science of observation an exact science, yet it enables us to apply our pure science, as far as it exhibits nature's laws the counterpart of our axioms, and thus to deduce from them the consequences which they involve. Again, the pure science includes *all laws* of communicated momentum, but as there is no *a priori* reason why nature's law should be one rather than another, observation and experiment are required to discover her particular law: here again the inductive process enters; and in the hands of Kepler it led to the three laws of equable description of areas, elliptic orbits, and sesquiplicate ratio of periodic times and mean distances; and these experimental laws having been ready for Newton to apply his pure science to, were by it translated into their mathematical equivalents, viz. the motion of planets under the action of a central force and in one plane, the law of the inverse square of the distance, and the action of the *same* central force on all the planets. Had not Kepler lived and enunciated his three laws, Newton might have constructed his fluxional calculus, and applied it to mechanics, and yet might not have enunciated the law of gravitation; a science of dynamics might have been formed, but it might have remained for some future observer to found the science of physical astronomy; numberless other instances have occurred in the science of mechanics of a similar kind; but what is above stated is enough to my present purpose to shew how insufficient experiment and induction is to establish the axiomatic laws of the pure science, and yet how necessary an examination of nature's facts is to enable us to apply our pure science to the explanation of phænomena of the external world.

An examination of the processes and methods of the still imperfect sciences, such as those of physical optics, heat, electricity, leads to the same results as that of the sciences of number, space, and motion.

In conclusion, I may remark that the preceding investigation

suggests the course to be pursued with the best prospect of success in an inquiry into the laws of nature; not only must the phænomena be examined to discover the necessary antecedent causes, and must she be subjected to crucial experiments, but the idea of colligation, which such an analysis has suggested, must be deductively traced and its results compared with the corresponding facts of observation; the two processes will thus mutually illustrate and strengthen each other, and finally a science will be formed with two phases, such as has been described in the preceding pages; it will have those notes of universality and necessity, which nothing short of the constitution of the human intellect can give to it, and it will also in part be useful as an applied science in unravelling the complicated wonders of the material world in which we are placed.